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Hamiltonian method and invariant search for 2D quadratic systems

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Abstract. The Hamiltonian formalism for a two-dimensional system of ODE possessing an invariant of the motion not containing the time explicitly has suggested a method for the search of first integrals. Applied to the quadratic system it leads to the finding of two phase space configurations.

1. Introduction

In recent years, much effort has been devoted to obtaining first integrals (invariants of the motion) for Hamiltonian and dynamical systems. The reason is that obtaining invariants corresponds to a partial integration and is interesting both from an analytical and numerical point of view. In this last point of view, obtaining one invariant is equivalent to reducing by one unit the dimension of the phase space. Since in nonlinear problems we are usually interested in a full exploration of initial conditions, any reduction of the dimension corresponds to a dramatic saving of numerical computation. Moreover, the existence of invariants provides, in complex problems, a welcomed check of the numerical scheme with respect to its accuracy and stability. In some cases [1], the knowledge of an invariant leads to the complete integration.

The problems treated here are those on population evolution (in biology, ecology and chemistry). A popular model investigated by many people—including the authors [2]—is the Lotka–Volterra (LV) one. Its generalization is the quadratic system (QS) used mostly in chemistry. For some QS we know particular invariants (see the work of Frommer [3] and Lunkevich and Sibirskii [4]).

Getting an invariant is not an easy task. Even if we consider only simple systems like LV or QS, general methods like that of Painlevé give only relations between the parameters of the equations but these relations are neither necessary nor sufficient in all cases and, moreover, no information is obtained on the form of the invariant. Up to now the most useful and practical method has been to assume a certain form for the invariant. Then, introducing the dynamical equations, we obtain an expression for the total derivative of the invariant. For simple systems this takes the form of a polynomial and subsequently all the coefficients of this polynomial are set to zero. Unfortunately, we have usually more equations than unkonwns (parameters of the invariant), and, thus, constraints, i.e. relations among the parameters of the equations

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must be introduced. The rule of the game is to have as few constraints as possible and consequently to take a form for the invariant as general as possible. For example the authors have managed to enlarge the set of known invariants for the LV system by taking an expression for the invariant I of the form (for a 2D system)

$$I = x^{\alpha} y^{\beta} P(x, y) \exp st \tag{1}$$

where P is a polynomial. The introduction of α and β allowed us to reduce the number of constraints to one. Obtaining this constraint α, β, and the polynomial P led to a set of linear algebraic systems and can be viewed as a generalization of the Carleman method. Unfortunately, the technique fails in the case of the QS since the relation $\alpha = \beta = 0$ is enforced.

An alternative to the above method is to assume a given form for the invariant and, through a rescaling technique, obtain the Hamiltonian and the general form of the equations which possess this kind of invariant. The resulting system of algebraic equations is now nonlinear but solutions seem possible.

If we remember that the LV system is a particular case of the QS

$$\begin{aligned} \frac{dx}{dt} &= a_1 x + b_{11} x^2 + b_{12} xy + c_1 y^2 \\ \frac{dy}{dt} &= a_2 y + b_{21} xy + b_{22} y^2 + c_2 x^2 \end{aligned} \tag{2}$$

where $c_1 = c_2 = 0$ we could expect to obtain three constraints for the QS. A remark should be made about the form of the linear part of (2): we choose to diagonalize the linear part and consequently a_1 and a_2 are just the eigenvalues of the linear matrix at the origin (eventually a_1 and a_2 are two complex conjugate numbers). Consequently (2) can be considered as the most general form. Here we assume all the equation parameters to be real.

We apply this Hamiltonian method to the QS and the paper is organized as follows: in section 2 we quickly introduce the method (given already in [5]) and we apply it to the case of an invariant of the form

$$I = P(x, y) [Q(x, y)]^{\mu}. \tag{3}$$

Since we will select for P a second-degree polynomial and for Q a first-degree one, we see that the case $\mu = 1$ must correspond to the third-degree polynomial type invariant obtained by Frommer [3]. Moreover, we consider in this section invariants of exponential type and show that they belong to the same family. In section 3 we will try an invariant of the form

$$I = P(x, y) [Q(x, y)]^{\mu} [R(x, y)]^{\nu} \tag{4}$$

where P, Q, R are three first-degree polynomials. It appears that (4) is equivalent for the QS to the form of invariant (1) given for the LV system. In section 4 we study the connection between invariant conditions and the marginal stability condition for equilibrium points. This connection is precised through a geometrical interpretation of the invariant conditions. Finally we present our conclusions in section 5.

2. The invariant PQ^{μ}

The method consists in predefining the type of the invariant and then adjusting the coefficients (direct method) in order to satisfy the differential equations. It is based on the Hamiltonian formalism of a system of two ordinary differential equations (ODE) possessing an invariant, which we recall here. Let

$$\frac{\mathrm{d}x}{\mathrm{d}t}=f(x,y) \qquad \frac{\mathrm{d}y}{\mathrm{d}t}=g(x,y) \tag{5}$$

be a system of two ODE having a time independent invariant. Here the system is taken naturally autonomous as the invariant is assumed to be time independent. In previous papers (see [1] or [5]) it was established that (5) can be written in a Hamiltonian form using a new time which is function of the phase space position:

$$\mathrm{d}\theta=F(x,y)\,\mathrm{d}t.$$

Note that this new time is in the same spirit as the one introduced by Hietarinta *et al* [6]. When a time-independent invariant $I(x,y)$ is known for system (5) then it can be written as follows

$$\frac{\mathrm{d}x}{\mathrm{d}\theta}=\frac{\partial I(x,y)}{\partial y} \qquad \frac{\mathrm{d}y}{\mathrm{d}\theta}=-\frac{\partial I(x,y)}{\partial x}$$

or equivalently in the old time t

$$\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{\partial I(x,y)}{\partial y}F(x,y) \qquad \frac{\mathrm{d}y}{\mathrm{d}t}=-\frac{\partial I(x,y)}{\partial x}F(x,y) \tag{6}$$

i.e. given a time independent invariant, $I(x,y)$ and given an arbitrary function $F(x,y)$ we can construct the most general two-dimensional ODE system for which $I(x,y)$ is a invariant. We use the property in the other way, in the sense that, assuming a form for $I(x,y)$ and given the system of ODE (5), we can select $F(x,y)$ and establish the algebraic equations to adjust the invariant constants.

Let us apply the method to the QS defined in (2). To start, consider an invariant of the type (3) with the following polynomials

$$P(x,y)=K+Ax+By+Cx^2+Dxy+Ey^2$$
$$Q(x,y)=1+\alpha x+\beta y$$

where K, A, B, C, D, E, α and β are constants, and apply our method. Then equations (6) become

$$\frac{\mathrm{d}x}{\mathrm{d}t}=\left(Q\frac{\partial P}{\partial y}+\mu P\frac{\partial Q}{\partial y}\right)FQ^{\mu-1} \qquad -\frac{\mathrm{d}y}{\mathrm{d}t}=\left(Q\frac{\partial P}{\partial x}+\mu P\frac{\partial Q}{\partial x}\right)FQ^{\mu-1}. \tag{7}$$

To obtain the two quadratic polynomials (i.e. the LHS of (2)), we must take $FQ^{\mu-1}$ to be a polynomial and the degree of the polynomials taken in (3) indicates that it must be simply a constant, i.e. $FQ^{\mu-1}=1$. Then we can construct the following table

$\mathrm{d}x/\mathrm{d}t$	$B+K\mu\beta$	$D+\alpha B$ $+\mu\beta A$	$2E+\beta B$ $+\mu\beta B$	αD $+\mu\beta C$	$2E\alpha+$ $(\mu+1)\beta D$	$(2+\mu)\beta E$
$-\mathrm{d}y/\mathrm{d}t$	$A+K\mu\alpha$	$2C+\alpha A$ $+\mu\alpha A$	$D+\beta A$ $+\mu\alpha B$	$(2+\mu)\alpha C$	$2C\beta+$ $(\mu+1)\alpha D$	$\beta D+\mu\alpha E$
coefficient of	1	x	y	x^2	xy	y^2

From the identification between this table and (2) one readily gets

$$\begin{aligned} A&=-K\mu\alpha\\ B&=-K\mu\beta\\ C&=K\mu(\mu+1)\alpha^2/2\\ E&=K\mu(\mu+1)\beta^2/2 \end{aligned} \tag{8}$$

and the diagonal linear terms (i.e. a_1x in $\dot{x}$ and a_2y in $\dot{y}$) give

$$D-K\mu(\mu+1)\alpha\beta=a_1=-a_2 \tag{9}$$

leading to the first constraint

$$a_1+a_2=0. \tag{10}$$

Now from the terms with c_1 and c_2 one obtains

$$\begin{aligned} \alpha^3K\mu(\mu+1)(2+\mu)/2&=-c_2\\ \beta^3K\mu(\mu+1)(2+\mu)/2&=c_1. \end{aligned} \tag{11}$$

D is computed identifying the coefficients of x^2 (in $\mathrm{d}x/\mathrm{d}t$) and y^2 (in $-\mathrm{d}y/\mathrm{d}t$) to b_{11} and $-b_{22}$ respectively

$$\begin{aligned} \alpha D+k\mu^2\frac{\mu+1}{2}\alpha^2\beta&=b_{11}\\ \beta D+K\mu^2\frac{\mu+1}{2}\alpha\beta^2&=-b_{22}. \end{aligned} \tag{12}$$

Finally, from the two last coefficients b_{12} and b_{21}, one obtains

$$\begin{aligned} K\mu(\mu+1)\alpha\beta^2+\beta D(\mu+1)&=b_{12}\\ K\mu(\mu+1)\alpha^2\beta+\alpha D(\mu+1)&=-b_{21}. \end{aligned} \tag{13}$$

Note that (11), (12) and (13) lead to two additional constraints:

$$\frac{b_{21}}{b_{12}}=\frac{b_{11}}{b_{22}}=\left(\frac{c_2}{c_1}\right)^{1/3}. \tag{14}$$

Moreover given b_{11}, b_{22}, c_1 and c_2, one obtains μ through one of the following:

$$b_{11}+b_{21}+c_1^{1/3}c_2^{2/3}=\mu(c_1^{1/3}c_2^{2/3}-b_{11})$$
$$b_{22}+b_{12}+c_2^{1/3}c_1^{2/3}=\mu(c_2^{1/3}c_1^{2/3}-b_{22}) \tag{15}$$

which are equivalent through (14). To obtain K let us multiply (9) by α

$$\alpha D-K\alpha^2\beta\mu(\mu+1)=\alpha a_1 \tag{16}$$

but as $K\alpha^2\beta$, αD and μ are K independent, α is determined from (16), K from (11) and consequently all other unknowns from (8) and (9). The results are as follows

$$\mu=\frac{b_{11}^2+b_{11}b_{21}+b_{22}c_2}{b_{22}c_2-b_{11}^2}$$
$$\alpha=-\frac{b_{21}b_{11}+2b_{22}c_2}{a_1b_{11}(\mu+1)}$$
$$\beta=-\alpha\frac{b_{22}}{b_{11}}$$
$$K=-\frac{2c_2}{\mu(\mu+1)(\mu+2)\alpha^3} \tag{17}$$
$$A=-K\alpha\mu$$
$$B=-K\beta\mu$$
$$C=\frac{\mu(\mu+1)K\alpha^2}{2}$$
$$D=K\mu(\mu+1)\alpha\beta+a_1$$
$$E=\frac{K\mu(\mu+1)\beta^2}{2}.$$

We note that in this problem, the number of equations (twelve) is equal to the sum of the number of unknowns (here they are nine, i.e. $K, A, B, C, D, E, \alpha, \beta, \mu$) and the number of constraints (three). Figure 1 is a typical phase portrait which can be described as follows. We assume the existence of four real equilibrium points. The equations (7) at equilibrium become

$$Q\frac{\partial P}{\partial y}+\mu P\frac{\partial Q}{\partial y}=0$$
$$Q\frac{\partial P}{\partial x}+\mu P\frac{\partial Q}{\partial x}=0 \tag{18}$$

or equivalently

$$P\left[\frac{\partial P}{\partial x}\frac{\partial Q}{\partial y}-\frac{\partial P}{\partial y}\frac{\partial Q}{\partial x}\right]=0$$
$$Q\left[\frac{\partial P}{\partial x}\frac{\partial Q}{\partial y}-\frac{\partial P}{\partial y}\frac{\partial Q}{\partial x}\right]=0.$$

Thus, at equilibria, either $P=0$, $Q=0$ or $[\partial P/\partial x\partial Q/\partial y-\partial P/\partial y\partial Q/\partial x]=0$. The last

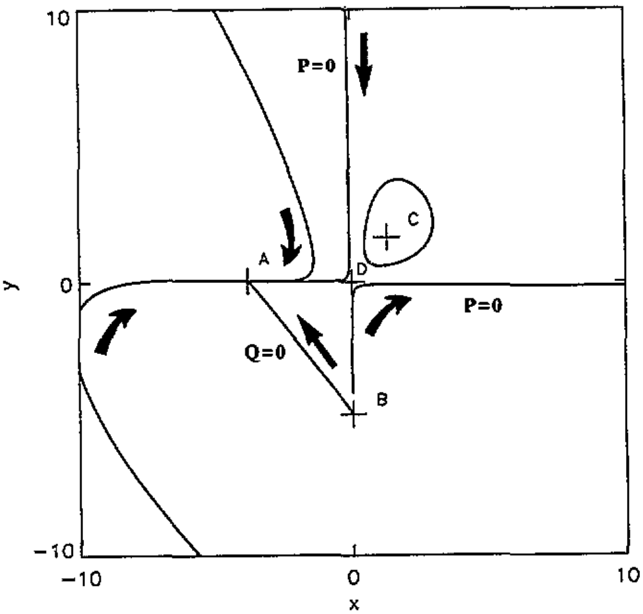

Figure 1. Phase portrait of a PQ^{μ} case: $a_1=-a_2=1$, $b_{11}=0.25$, $b_{12}=-0.8$, $b_{21}=1$, $b_{22}=-0.2$, $c_1=-0.008$, $c_2=0.015625$.

case leads to $(A+Dy+2Cx)\beta-(B+Dx+2Ey)\alpha=0$, a linear relation between x and y, which when substituted into (18) gives the origin and one other equilibrium point. The other two equilibrium points are then obtained from the intersection of $P=0$ (a conic section) and $Q=0$ (a line). Since for $P=0$, $I=0$, and for $Q=0$, $I=0$ or infinity (depending on the sign of μ), then $P=0$ and $Q=0$ are trajectories of the system.

2.1. *The Frommer invariant*

The particular case $\mu=1$ corresponds to the Frommer invariant [3]. It is straightforward to see that using the results (17) that the invariant expression (3) becomes

$$I=a_1xy-\frac{c_2}{3}x^3+b_{11}x^2y-b_{22}xy^2+\frac{c_1}{3}y^3$$

with the conditions

$$a_1+a_2=2b_{11}+b_{21}=2b_{22}+b_{12}=0. \tag{19}$$

The two last relations are deduced directly from (15) putting $\mu=1$. If $c_1=c_2=0$ the QS coincides with the LV system and the Frommer invariant is given by

$$I=xy(a_1+b_{11}x-b_{22}y).$$

This is a particular case of the invariant III for LV systems [2] as the first constraint of (14) together with $a_1+a_2=0$ implies $R_{12}\equiv(a_1+a_2)b_{11}b_{22}-a_1b_{21}b_{22}-a_2b_{12}b_{11}=0$, which is the unique constraint required for this invariant. The general form has been obtained through the Carleman method as mentioned previously.

2.2. *Invariants of the type P exp (Q)*

Similarly we can search invariants of the type

$$I(x,y)=P(x,y)\exp Q(x,y) \tag{20}$$

where P and Q are polynomials. Then the dynamical system satisfies

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t}&=\left(\frac{\partial P}{\partial y}+P\frac{\partial Q}{\partial y}\right)F\exp Q\\ -\frac{\mathrm{d}y}{\mathrm{d}t}&=\left(\frac{\partial P}{\partial x}+P\frac{\partial Q}{\partial x}\right)F\exp Q\end{aligned} \tag{21}$$

and $F\exp(Q)$ must be a polynomial. Assuming that P and Q are, respectively, of first and second degree

$$\begin{aligned}P(x,y)&=K+Ax+By\\ Q(x,y)&=\alpha x+\beta y+\delta x^2+\varepsilon xy+\eta y^2\end{aligned} \tag{22}$$

then $F\exp(Q)$ can be taken constant (i.e. $=1$). Replacing in (21) the polynomial expressions (22), we obtain in a straightforward manner

$$\begin{aligned}K&=a_1^3(c_1^{1/3}c_2^{1/3}-b_{11}c_{21}^{-1/3})^3\\ \varepsilon K^{2/3}&=-b_{11}/c_2^{1/3}=-b_{22}/c_1^{1/3}\\ \alpha^3&=c_2/K\\ \beta^3&=-c_1/K\\ \eta&=\beta^2/2\\ \delta&=\alpha^2/2\\ A&=-\alpha K\\ B&=-\beta K.\end{aligned} \tag{23}$$

Now $a_1+a_2=0$ is still required and as can be seen from (23) we get

$$\frac{b_{11}}{b_{22}}=\left(\frac{c_2}{c_1}\right)^{1/3}. \tag{24}$$

However, two additional constraints among the coefficients must hold, namely

$$\begin{aligned}b_{12}&=-(c_1^{2/3}c_2^{1/3}+b_{22})\\ b_{21}&=-(c_1^{1/3}c_2^{2/3}+b_{11}).\end{aligned} \tag{25}$$

In summary there are four relations among the coefficients, and we note that the PQ^{μ} constraints (14) are fullfilled.

Consider now

$$\begin{aligned}P(x,y)&=K+Ax+By+Cx^2+Dxy+Ey^2\\ Q(x,y)&=\alpha x+\beta y\end{aligned} \tag{26}$$

then, the invariant coefficients satisfy

$$K=a_1^3\left[\frac{b_{12}}{(2c_1)^{1/3}}+(4c_1c_2)^{1/3}\right]^{-3}$$

$$\alpha^3 = -2c_2/K$$
$$\beta^3 = 2c_1/K$$
$$A = -\alpha K$$
$$B = -\beta K$$
$$C = \alpha^2 K/2$$
$$E = \beta^2 K/2$$
$$D - \alpha\beta K = a_1 = -a_2$$

with the constraint $b_{11}/b_{22} = b_{21}/b_{12} = -\alpha/\beta$ and the new ones

$$\begin{aligned} b_{11} &= c_1^{1/2} c_2^{2/3} \\ b_{22} &= c_1^{2/3} c_2^{1/3}. \end{aligned} \tag{27}$$

We see that the relations (14) are again satisfied. As a consequence we notice that both exponential invariant satisfy four conditions, i.e. one more that the invariant PQ^μ. However, all three conditions ((10) and (14)) of this invariant are satisfied by the exponential invariant meaning that this invariant belongs to the same family. One can see, in fact, that this increase in the number of constraints originates in a degeneracy of μ, i.e. the full conditions (25) or (27) are obtained as limit cases. Let us consider first in (15) $\mu \to \infty$. We immediately see that then we fall on conditions (27). On the other hand, if we take $\mu \to 0$, we obtain directly the conditions (25). We can explain now the reason for the presence of the fourth constraint in the exponential invariant as μ disappears; the number of unknowns is lowered by one.

Returning again to the LV system (case $c_1 = c_2 = 0$), one can search for invariants of type (20). What results is the Volterra invariant [7] as a particular case of (20) with (26). In fact one can immediately show that the constraints are now (10) and $b_{11} = b_{22} = 0$ and the invariant is given by

$$I = xy \exp(b_{21}x - b_{12}y). \tag{28}$$

3. Invariants of the type $PQ^\mu R^\nu$

Let us consider now an invariant of type (4), where P, Q, R are the following linear polynomials

$$\begin{aligned} P &= 1 + \alpha x + \beta y \\ Q &= 1 + Ax + By \\ R &= 1 + Cx + Dy. \end{aligned} \tag{29}$$

The equations (6) are

$$\begin{aligned} \frac{dx}{dt} &= \left(QR\frac{\partial P}{\partial y} + \mu PR\frac{\partial Q}{\partial y} + \nu PQ\frac{\partial R}{\partial y} \right) T \\ -\frac{dy}{dt} &= \left(QR\frac{\partial P}{\partial x} + \mu PR\frac{\partial Q}{\partial x} + \nu PQ\frac{\partial R}{\partial x} \right) T \end{aligned} \tag{30}$$

with $T = FQ^{\mu-1}R^{\nu-1}$. Then the QS will require T=constant. Here we must take a constant since all the polynomial P, Q, R begin with 1 as the constant term in contradiction to the preceding case. Another point of view is to divide all the a_i and b_{ij} by this constant remembering that we still have nine constants at our disposal. From now the a_i and b_{ij} wil be defined within a constant (the same for all coefficients). The result of equating (30) to (2) is the following set of equations: first, the constant terms give

$$\beta + \mu B + \nu D = 0 \tag{31}$$

$$\alpha + \mu A + \nu C = 0. \tag{32}$$

The linear diagonal terms contribute with

$$\beta(C + A) + \mu B(\alpha + C) + \nu D(\alpha + A) = a_1 \tag{33}$$

$$\alpha(D + B) + \nu C(\beta + B) + \mu A(\beta + D) = -a_2. \tag{34}$$

The linear non-diagonal terms are

$$\alpha(C + A) + \mu A(\alpha + C) + \nu C(\alpha + A) = 0 \tag{35}$$

$$\beta(D + B) - \nu D(\beta + B) + \mu B(\beta + D) = 0. \tag{36}$$

The terms on b_{11} and b_{22} are, respectively

$$\alpha\mu CB + \beta AC + \alpha\mu AD = b_{11} \tag{37}$$

$$\beta\nu BC + \alpha BD + \beta\mu AD = -b_{22}. \tag{38}$$

The terms on c_1 and c_2 are, respectively

$$\beta(1 + \mu + \nu)DB = c_1 \tag{39}$$

$$\alpha(1 + \mu + \nu)AC = -c_2. \tag{40}$$

Finally the terms on b_{12} and b_{21} can be written

$$\beta(BC + AD) + \mu B(\beta C + \alpha D) + \nu D(\beta A + \alpha B) = b_{12} \tag{41}$$

$$\alpha(BC + AD) + \mu A(\beta C + \alpha D) + \nu C(\beta A + \alpha B) = -b_{21}. \tag{42}$$

The strategy to solve the system (31) to (42) is first to replace the values of α, β from (31) and (32). So doing, (35) and (36) transform to

$$(\nu C + \mu A)^2 + \mu A^2 + \nu C^2 = 0 \tag{35a}$$

$$(\mu B + \nu D)^2 + \mu B^2 + \nu D^2 = 0. \tag{36a}$$

As a consequence at least one of the invariant's exponents μ or ν is negative. Moreover the ratios B/D and A/C satisfy the same quadratic equation. Let us call

$$\frac{B}{D} = m \qquad \frac{A}{C} = p. \tag{43}$$

Then (33) and (34) can be written as

$$\frac{\mu\nu^2}{1+\mu} - \nu(1+\nu) = -\frac{a_1}{2CD} \tag{33a}$$

$$\frac{\mu\nu^2}{1+\mu} - \nu(1+\nu) = \frac{a_2}{2CD} \tag{34a}$$

giving again condition (10). Moreover m and p are the roots of the second-degree equations (35a) or (36a) which coincide. Hence we have

$$m+p=-\frac{2\nu}{1+\mu} \qquad mp=\frac{\mu(1+\nu)}{\mu(1+\mu)}. \tag{44}$$

We use (44) to simplify the left-hand sides of equations (41) and (42). Then one can see easily that these equations coincide with (35a) and (36a), leading to the constraints

$$b_{12}=b_{21}=0. \tag{45}$$

The remaining equations transform to

$$p^2\mu\nu+pm\mu(1+\mu)+p\nu(1+\nu)+m\mu\nu=-\frac{b_{11}}{C^2D} \tag{37a}$$

$$m^2\mu\nu+mp\mu(1+\mu)+m\nu(1+\nu)+p\mu\nu=\frac{b_{22}}{CD^2} \tag{38a}$$

$$(1+\mu+\nu)(\mu p+\nu)p=\frac{c_2}{C^3} \tag{39a}$$

$$(1+\mu+\nu)(\mu m+\nu)m=-\frac{c_1}{D^3}. \tag{40a}$$

One can eliminate from these the quotient C/D and obtain an equation which, together with (44), will lead to $\nu=\Phi(\mu)$. Then (33a), (37a) and (38a) will determine ν, C and D and the problem is solved, although the formulation is rather cumbersome. Note that here the number of knowns was eight and the number of equations is reduced to eleven as pointed out before. This is consistent with the number of constraints found which is three.

The phase portrait of one example of this invariant is presented in figure 2. The main characteristics are the existence of trajectories defining a triangle joining the three equilibrium points which are not the origin. In contrast with the PQ^μ case, here there is no centre. Assuming the existence of four real equilibrium points, figure 2 could be described as follows. At equilibria, equations (30) become

$$\begin{aligned} QR\frac{\partial P}{\partial y}+\mu PR\frac{\partial Q}{\partial y}+\nu PQ\frac{\partial R}{\partial y}=0 \\ QR\frac{\partial P}{\partial x}+\mu PR\frac{\partial Q}{\partial x}+\nu PQ\frac{\partial R}{\partial x}=0 \end{aligned} \tag{46}$$

or, equivalently,

$$P\left[\mu R\left(\frac{\partial Q}{\partial y}\frac{\partial P}{\partial x}-\frac{\partial Q}{\partial x}\frac{\partial P}{\partial y}\right)+\nu Q\left(\frac{\partial R}{\partial y}\frac{\partial P}{\partial x}-\frac{\partial R}{\partial x}\frac{\partial P}{\partial y}\right)\right]=0. \tag{47}$$

Thus at equilibria either $P=0$ or the term in the brackets must cancel. The case for which this last assumption is true gives $R=Q$ after taking into account the definitions (29) in the derivatives and using (31) and (32). The relation $R=Q$, when substituted into (46), leads to the origin and $R=0$, hence $Q=0$, as equilibria. Now the case $P=0$ as equilibrium along with $R=0$ and $Q=0$ produce three intersection points. Note that

(31), (32) and (43) imply that $P=0$, $Q=0$ and $R=0$ are not parallel lines. Thus their intesection must be three points. Also for $P=0$, $I=0$, for $Q=0$, $I=0$ or infinity (depending on the sign of μ), and for $R=0$, $I=0$ or infinity (depending on the sign of ν), hence the lines $P=0$, $Q=0$ and $R=0$ are trajectories of the system (2). Note that assuming $m=p$ would lead from (44) to $1+\mu+\nu=0$ and to the cancellation of all the QS coefficients.

4. Connection between marginal stability and existence of an invariant

After establishing the generality of invariant conditions (14), we examine the relations between invariant conditions and the nature of the equilibrium points. Let us call x_0, y_0 the coordinates of an equilibrium point. These satisfy the equations

$$\begin{aligned} &a_1x_0+b_{11}x_0^2+b_{12}x_0y_0+c_1y_0^2=0\\ &a_2y_0+b_{21}x_0y_0+b_{22}y_0^2+c_2x_0^2=0. \end{aligned} \tag{48}$$

The linearization of the QS around this point ($x=x_0+\varepsilon_x$, $y=y_0+\varepsilon_y$) leads to

$$\begin{aligned} &\dot{\varepsilon}_x=(a_1+2b_{11}x_0+b_{12}y_0)\varepsilon_x+(b_{12}x_0+2c_1y_0)\varepsilon_y\\ &\dot{\varepsilon}_y=(b_{21}y_0+2c_2x_0)\varepsilon_x+(a_2+2b_{22}y_0+b_{21}x_0)\varepsilon_y. \end{aligned} \tag{49}$$

Let us recall here that among the solutions that a linear system like (49) can have, there are those which are neither amplified nor dumped (marginal stable) corresponding to the case where the eigenvalues are purely imaginary (case of a centre). This implies that the sum of the eigenvalues is zero. We say that this is the condition for the

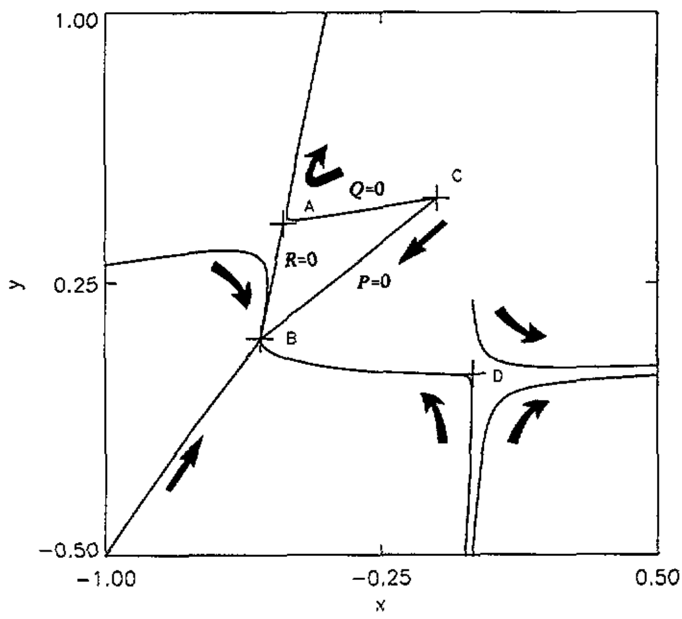

Figure 2. Phase portrait of a $PQ^{\mu}R^{\nu}$ case: $a_1=1.634$, $a_2=-1.634$, $b_{11}=2.798$, $b_{12}=b_{21}=0$, $b_{22}=3.362$, $c_1=0.5635$, $c_2=0.37566$. The invariant coefficients are $A=0.333$, $B=-2$, $C=1.666$, $D=-0.333$, $\alpha=1.4253$, $\beta=-1.7816$, $\mu=-0.77408$, $\nu=-0.70036$.

marginal stability. But under this definition we include the eigenvalues λ and $-\lambda$ where λ is real, i.e. the saddle points, which play the role of imaginary centers. This must be checked afterwards. For (49), the marginal stability condition can be written

$$S=a_1+a_2+(2b_{11}+b_{21})x_0+(2b_{22}+b_{12})y_0=0. \tag{50}$$

We can ask then whether the marginal stability is a signature for the existence of an invariant. To answer this question we note first that the invariant constraint (10), namely $a_1+a_2=0$, forces the origin to satisfy $S=0$. We are going to find now that for the invariants of the PQ^{μ} family only, the other two constraints lead to another marginaly stable equilibrium point. This is immediate for the Frommer case from inspection of constraints (19). For the general PQ^{μ} invariant it is also true. Let k be the ratio appearing in (14)

$$\frac{b_{21}}{b_{12}}=\frac{b_{11}}{b_{22}}=\left(\frac{c_2}{c_1}\right)^{1/3}=k. \tag{51}$$

Consequently we can write $b_{11}x_0=kb_{22}x_0$, $b_{21}x_0=kb_{12}x_0$, $c_2(x_0^2/y_0)=k^3c_1x_0^2/y_0$. Now taking into account (10) and adding the two equations (48) divided, respectively, by x_0 and y_0, we obtain

$$b_{22}(y_0+kx_0)+b_{12}(y_0+kx_0)+\left(\frac{c_1}{x_0y_0}\right)(y_0^3+k^3x_0^3)=0. \tag{52}$$

Now (52) indicates simply that there exist equilibrium points satisfying the relation $y_0+kx_0=0$. One of these being the origin, the other is obtained explicitly through the value of k from (51). The equation to be satisfied by this equilibrium point is $(2b_{11}+b_{21})x_0+(2b_{22}+b_{12})y_0=0$, which, with (10), is the condition of marginal stability (50). Outside of the origin there will be another equilibrium point satifying condition (50) (point C of figure 1). This point will be saddle or centre depending on whether the eigenvalues for it are real or complex. Moreover, according to the theorem of Kukles and Casanova [8] reported by Coppel [9], if the quadrilateral with vertices at the equilibrium points is convex, then there are two opposite saddles and so the two points considered before are two opposite vertices. If the quadrilateral is not convex then either the three exterior vertices are saddles and the interior is an *antisaddle (nodes, foci or centres) or the exterior vertices are antisaddles and the* interior is a saddle. Figure 1 is an example of this last case, the antisaddles being respectively a stable node (A), an unstable node (B) and a centre (C). The constraints (45) do not satisfy (50) in general and consequently for invariant $PQ^{\mu}R^{\nu}$, only the origin is marginaly stable.

5. Conclusion

The method used in this paper is based on the fact that a 2D system possessing a time-independent invariant can be put in a Hamiltonian form through rescaling, the Hamiltonian being the invariant. As in most working methods we must assume a certain form for the introduced invariant and we build the general equations of the systems having this invariant. We have an arbitrary function used to cancel difficult terms which would lead to systems of equations different from those we intend to study. This method plays, for the QS, the role of the term $x^{\alpha}y^{\beta}$ which we previously introduced in the LV system using then the extended Carleman method (which does not work in the QS case). As a matter of fact the type III invariant that we found in the LV system is the counterpart of the invariant of the form $PQ^{\mu}R^{\nu}$ studied in section 3. In this last case $P=Q=R=0$ describes three sides of the triangle formed by the three equilibrium points excluding the origin. In the same way for LV invariants of type $x^{\alpha}y^{\beta}P(x, y)$ where P is a first-degree polynomial, $x=y=P=0$ describes the three sides of the triangle formed by the origin and two equilibrium points on the coordinate axes. As a consequence we can forecast for the QS other invariants built on other triangles. To obtain them we must omit the constant term 1 in the assumed form of Q and R introduced in section 3. It will give a small new result but in the 'invariant fishing' business one should be patient and accept little fishes!

We next consider what other generalizations are in store. For the moment we see two roads. The first is the study of time-dependent invariants. In fact the search for an invariant of the form $I(x, y, t)=J(x, y)\exp(st)$ can be associated with the search for explicitly time-independent invariant $J(x, y)$ if we have $a_1=a_2$. This explains the type II family found for the LV system. Generalization to higher order polynomials will probably be easy.

The relation $a_1=a_2$ is interesting. In fact we know that for the QS, without any further constraint, a rescaling method leads to a solution through a cascade of quadratures (see appendix). In our opinion this clearly indicates that the problem in searching for an invariant is not so much their existence but the explicit form that we are seeking for them. These results will be published latter.

The second—and more exciting question—is the generalization to a higher dimension. The first step is, of course, to a 3D system. As the number of equations is odd no Hamiltonian can be exhibited. However, one can write the general form of the equations for a system having an invariant $I(x, y, z)$ using three arbitrary functions. For higher dimensions we face the problem of too many constraints when we assume a given form for the invariant. Of course too many constraints take out the usefulness of the invariant.

Since we have now a good knowledge of the invariants for both the 2D LV and QS, it will be interesting to compare with the Painlevé method. What comparison can be made between the cases given by this method and the ones given by Painlevé? Does the Painlevé method point out cases not given by the Carleman or the Hamiltonian schemes? What is the form of the invariants where Painlevé indicates there is one—and when has it been found by the methods we proposed?

Symmetry of a system, existence of invariants, integrability are different—but intimately connected—concepts. Exhaustive and systemtaic studies of the few systems where it is possible are crucial for a clear understanding of these topics. This paper is a contribution to this task.

Appendix. Integrability of n-homogeneous polynomial systems in 2D

Let us consider the system

$$\frac{dx}{dt}=a_1x+f(x, y)$$
$$\frac{dy}{dt}=a_2y+g(x, y) \tag{A1}$$

where $f(x, y)$ and $g(x, y)$ are homogeneous polynomials of degree n, which includes the *qs* (2) as the particular case $n=2$. Assume moreover that

$$a_1=a_2. \tag{A2}$$

Then we can apply to this system a rescaling similar to the one introduced by Coste–Peyraud–Coullet [10]. This allows us to obtain the solution of (A1) in an elegant manner.

Theorem. If $a_1=a_2=a$ one can solve the n-degree polynomial ODE (A1) by the following rescaling

$$x=\lambda(t)\xi_1 \qquad y=\lambda(t)\xi_2 \qquad \mathrm{d}\theta=\mu\,\mathrm{d}t \tag{A3}$$

Proof. It is easy to see that system (A1) with condition (A2) transforms under the rescaling (A3) in

$$\begin{aligned}\lambda\mu\frac{\mathrm{d}\xi_1}{\mathrm{d}\theta}&=(a\lambda-\dot{\lambda})\xi_1+\lambda^n f(\xi_1\xi_2)\\ \lambda\mu\frac{\mathrm{d}\xi_2}{\mathrm{d}\theta}&=(a\lambda-\dot{\lambda})\xi_2+\lambda^n g(\xi_1\xi_2)\end{aligned} \tag{A4}$$

where $\dot{\lambda}=\mathrm{d}\lambda/\mathrm{d}t$. Take

$$\mu=\lambda^{n-1} \tag{A5}$$

and introduce

$$K(\theta)=\frac{a\lambda-\dot{\lambda}}{\lambda^n}$$

which we transform to a differential equation in $\lambda(\theta)$ (i.e. λ expressed as a function of the new time θ)

$$\frac{\mathrm{d}\lambda}{\mathrm{d}\theta}+K(\theta)\lambda=a\lambda^{2-n}. \tag{A6}$$

Then (A4) may be written

$$\begin{aligned}\frac{\mathrm{d}\xi_1}{\mathrm{d}\theta}&=K(\theta)\xi_1+f(\xi_1\xi_2)\\ \frac{\mathrm{d}\xi_2}{\mathrm{d}\theta}&=K(\theta)\xi_2+g(\xi_1\xi_2).\end{aligned} \tag{A7}$$

We can make a choice of $K(\theta)$ such that $\mathrm{d}\xi_2/\mathrm{d}\theta=0$. Under these conditions we have $\xi_2=C$. If ξ_1 is known, so is $K(\theta)$. We can set the constant equal to unity without any loss of generality, as can be shown easily. Then from the second of equations (A7)

$$K(\theta)=-g(\xi_1 1)$$

and the first of (A7) can be written

$$\mathrm{d}\theta=\frac{\mathrm{d}\xi_1}{-g(\xi_1 1)\xi_1+f(\xi_1 1)} \tag{A8}$$

where the variables separate and, consequently, the solution is reduced to a quadrature. The knowledge of ξ_1 can then be used to obtain λ through (A6). Put

$$\phi(\theta)=\int_0^\theta K(\theta')\,\mathrm{d}\theta' \qquad \lambda(\theta)=A(\theta)\,\mathrm{e}^{-\phi(\theta)}. \tag{A9}$$

One has

$$\frac{\mathrm{d}A}{\mathrm{d}\theta}\,\mathrm{e}^{-\phi(\theta)}=aA^{2-n}\,\mathrm{e}^{0(n-2)\phi(\theta)}$$

which integrates as

$$A=\left\{a(n-1)\int_0^\theta \mathrm{e}^{(n-1)\phi(\theta')}\,\mathrm{d}\theta'+\lambda_0^{n-1}\right\}^{1/n-1} \tag{A10}$$

where we have introduced the value λ_0 of λ at $t=\theta=0$ (the two times coincide at the origin). The value of μ is then

$$\mu(\theta)=\mathrm{e}^{-(n-1)\phi(\theta)}\left[a(n-1)\int_0^\theta \mathrm{e}^{(n-1)\phi(\theta')}\,\mathrm{d}\theta'+\lambda_0^{n-1}\right]. \tag{A11}$$

The problem is then reduced to a sequence of quadratures, the last being the one which establish the correspondence $t\leftrightarrows\theta$ through (A3).

The unusual character of this rescaling must be pointed out. Note that here we introduce the scales λ and μ (for x and time) and solve the new variable (although we ignore these scales). The important property is that this choice allows one to eliminate formally the second dependent variable. The possibility of obtaining the equation for the scale from the solution $\xi_1(\theta)$ is given by the fact that in (A7) it is the same $K(\theta)$ which appears—with the only condition $a_1=a_2$. Moreover, the subsequent equation (A6) although nonlinear can be solved (in contrast with many problems where the difficulties eliminated in the beginning reappear in the last step!). This interesting property of 2D systems was also apparent in LV equations [2], where indeed an invariant was obtained for $a_1=a_2$.

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